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RECONSTRUCTION OF THE SHAPE OF A CONVEX DEFECT FROM A SCATTERED WAVE FIELD IN THE RAY APPROXIMATION†

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The problem of reconstructing the shape of a convex defect in a solid body from the results of measurements of the amplitude of back-scattering of an ultrasound wave is considered. It is assumed that the length-scale of the defect is much larger than the wavelength, which allows the problem to be considered in the ray approximation. It has been shown [1] that, using such an approach, the problem being investigated reduces to the well-known Minkowski problem: for a given Gaussian surface curvature reconstruct the shape of a closed convex surface. It has been shown [2] that under specified conditions a unique convex surface exists which has the Gaussian curvature of a given continuous function. However, an algorithm allowing one to construct such a convex surface does not exist. In this paper a numerical method is developed which enables one to implement the reconstruction of the required convex surface. As examples we consider the reconstruction of ellipsoids of revolution with various eccentricities, and also a nearly-cylindrical surface.

1. SUPPOSE there is a defect in an elastic medium in the form of a cavity of unknown shape. One of the most widely used methods of detecting and identifying such defects consists of using non-destructive ultrasonic monitoring. It involves irradiating the unknown boundary of the defect, which is a closed surface S, with plane ultrasonic waves at all possible angles of incidence, and measuring the amplitudes of the reflected echo-waves in the far field. In this paper we will restrict ourselves to the case when the unknown surface S is smooth and convex. One can show that under irradiation, in the echo regime, using existing ultrasonic transducers, one of the two types of wave (longitudinal or transverse) will always predominate over the other [3]. Because of this it can be shown that the amplitude of the reflected wave in the ray approximation has the same form, as in the acoustic case (i.e. in a medium where only one of these types of wave can propagate). This amplitude can be expressed in terms of the Gaussian curvature of the surface at the point at which the direction of the normal coincides with the direction of the radiation [1]:

$$A=c\sqrt{\gamma},\quad \gamma=R_1R_2 \tag{1.1}$$

where c is some constant. Thus, in the present approach, the problem being investigated reduces to the Minkowski problem, well-known in differential geometry: from a given Gaussian curvature reconstruct the shape of a smooth closed convex surface [2, 4].

Later we shall require the following theorem.

Theorem 1 [2]. Suppose $\gamma(\alpha)$ is a positive continuous function such that

$$\int\limits_{\Omega}\gamma(\alpha)\bar{\mathbf{n}}(\alpha)\,d\alpha=0 \tag{1.2}$$

where Ω is the total solid angle of three-dimensional space. Then there exists a unique smooth closed convex surface for which $\gamma^{-1}(\alpha)=(R_1R_2)^{-1}$ for an arbitrary $\alpha\in\Omega$ is the Gaussian curvature at the point with normal $\mathbf{n}(\alpha)$.

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A specific construction of the required surface can be implemented by various methods. The most efficient one is based on the use of the concept of the support function.

We will introduce an auxiliary function $P(\alpha)$ [4] connected as follows, with the distance $p(\alpha)$ from the origin of coordinates to the tangent plane with normal α:

$$P(\alpha_1, \alpha_2, \alpha_3) = rp(\alpha_1/r, \alpha_2/r, \alpha_3/r)$$
$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad r = (\alpha_1^2+\alpha_2^2+\alpha_3^2)^{1/2} \tag{1.3}$$

Here α is the exterior normal vector to the surface. The function $P(\alpha)$ satisfies the following relation [4]:

$$P_{11}P_{22}+P_{11}P_{33}+P_{22}P_{33}-P_{12}^2-P_{13}^2-P_{23}^2=\gamma(\alpha)$$
$$(P_{ij}=\partial^2 P/\partial\alpha_i\partial\alpha_j; \quad i, j=1, 2, 3) \tag{1.4}$$

where $\gamma(\alpha)>0$ is the reciprocal of the known Gaussian curvature. Thus the problem reduces to solving the non-linear partial differential equation (1.4).

2. Unlike the problem investigated in the present paper, the original problem of reconstructing a convex surface from the mean curvature reduces to a linear partial differential equation for the function $P(\alpha)$, the solution of which is constructed by a spherical harmonic expansion. Apart from the fact that this approach cannot be applied to a non-linear equation of the form (1.4), it is also inefficient for essentially non-spherical surfaces. In this paper we propose a completely different method.

We change from Cartesian to spherical coordinates in Eq. (1.4), using the relations

$$P_1=P_r \sin\theta \cos\varphi+P_\theta \cos\theta \cos\varphi/r-P_\varphi \sin\varphi/(r\sin\theta) \tag{2.1}$$
$$P_2=P_r \sin\theta \sin\varphi+P_\theta \cos\theta \sin\varphi/r+P_\varphi \cos\varphi/(r\sin\theta)$$
$$P_3=P_r \cos\theta-P_\theta \sin\theta/r \quad (P_i=\partial P/\partial\alpha_i, \quad i=1, 2, 3)$$

One similarly calculates the second derivatives P_{ij} $(i, j = 1, 2, 3)$.

Using well-known relations [4] governing the properties of the function $P(\alpha)$, we have

$$\partial P/\partial r=P/r, \quad \partial P_i/\partial r=0 \quad (i=1, 2, 3) \tag{2.2}$$

The original equation (1.4) holds throughout the space of the variables α_1, α_2, α_3. The transition to spherical coodinates using relations (2.2) enables one to get rid of the derivatives with respect to r. This makes it sufficient to solve the resulting non-linear differential equation on the unit sphere (with $r = 1$).

To solve the given equation on the unit sphere one uses difference expressions for the first and second derivatives on a grid consisting of parallels and meridians. Taking grid steps for θ $(0\leqslant\theta_i\leqslant\pi)$ and ϕ $(0\leqslant\phi_j\leqslant 2\pi)$ that are identical and equal to $h = \pi/N$, we have at the (i, j) node

$$P_\theta=(P^{i+1,j}-P^{i-1,j})/(2h), \quad P_\varphi=(P^{i,j+1}-P^{i,j-1})/(2h)$$
$$P_{\theta\theta}=(P^{i+1,j}-2P^{i,j}+P^{i-1,j})/h^2, \quad P_{\varphi\varphi}=(P^{i,j+1}-2P^{i,j}+P^{i,j-1})/h^2 \tag{2.3}$$
$$P_{\theta\varphi}=(P^{i+1,j+1}-P^{i-1,j+1}-P^{i+1,j-1}+P^{i-1,j-1})/(4h^2)$$

Substituting relations (2.1) and (2.3) into (1.4) we obtain the non-linear system of algebraic equations

$$AP=\gamma \tag{2.4}$$

In order to obtain a one-dimensional vector of unknowns P^k on the two-dimensional grid P^{ij} it is necessary to renumber the nodes (θ_i, ϕ_j) in the form of a one-dimensional array. The following node-numbering procedure is proposed. For each ϕ_j the sequential numbering corresponds to increasing angle θ_i: $h\leqslant\theta_i\leqslant\pi-h$. The process then continues to the next higher value ϕ_{j+1}. The last two nodes are the two poles. Finally the dimensions of system (2.4) are $2N(N-1)+2$. The Newton–Kantorovich method is applied to this system, in which the subsequent approximation P_{n+1} is expressed in terms of P_n in the following manner:

$$P_{n+1}=P_n+z_n \tag{2.5}$$

where z_n is found from the linear algebraic system

$$Jz_n=\gamma-AP_n \tag{2.6}$$

Here $J = A'$ is the Jacobian of system (2.4). The structure of the very sparse matrix J is as follows: there is a diagonal band of half-width N and there are two final rows and two final columns. The linearized system (2.6) is therefore not a system with a banded matrix. Consequently, the method of solving it using the sparseness of the matrix J is much more complicated than a simple pivotal condensation.

We used well-known methods for solving sparse system [5]. Practical implementation showed good convergence of the proposed approach for all the convex surfaces considered. Here the computing time on an IBM PC AT did not exceed 30 s.

The method considered in this paper for reconstructing the form of a convex defect, applicable in the case of very short waves, enables one to reach an interesting physical conclusion.

We assume that in this short-wave range the amplitude of the back-scattering $A(\alpha)$ is known for an arbitrary non-convex imperfection $[A(\alpha)>0]$. Such a function is of course continuous everywhere [6]. We introduce an auxiliary function γ defined by formulas (1.1):

$$\gamma(\alpha)=A^2(\alpha)/c^2 \tag{2.7}$$

The result of Theorem 1 says that when condition (1.2) is satisfied there exists a convex surface with Gaussian curvature $1/\gamma$. It follows that in the context of the short-wave asymptotic form we have the following theorem.

Theorem 2. Suppose $A(\alpha)$ is the amplitude of back-scattering from an arbitrary object. If

$$\int_\Omega \bar{\mathbf{n}}(\alpha)A^2(\alpha)\,d\alpha=0 \tag{2.8}$$

then as well as the real object there also exists a convex surface whose reflection diagram is identical with the function $A(\alpha)$.

Corollary. If the actual defect is symmetrical about the origin of coordinates, the result of Theorem 2 is automatically valid.

Indeed, in this case integral (2.8) is always equal to zero, because the symmetry means that the positive and negative contributions to the integrals mutually cancel. We note that the symmetry of the original object about the coordinate axes is a special case of symmetry about the origin of coordinates.

Figure 1 shows two-dimensional contours that cannot be distinguished at high frequencies, 2λ is the characteristic size of the real object and λ is the wavelength ($\lambda=2\pi$).

3. In the two-dimensional case the amplitude of the scattered wave can be expressed by [1]

$$A=1/2\sqrt{\pi k\rho(\alpha)} \tag{3.1}$$

where k is the wave number. In this case the problem consists of finding a convex contour l with curvature $[\rho(\alpha)]^{-1}$ $(\rho>0)$.

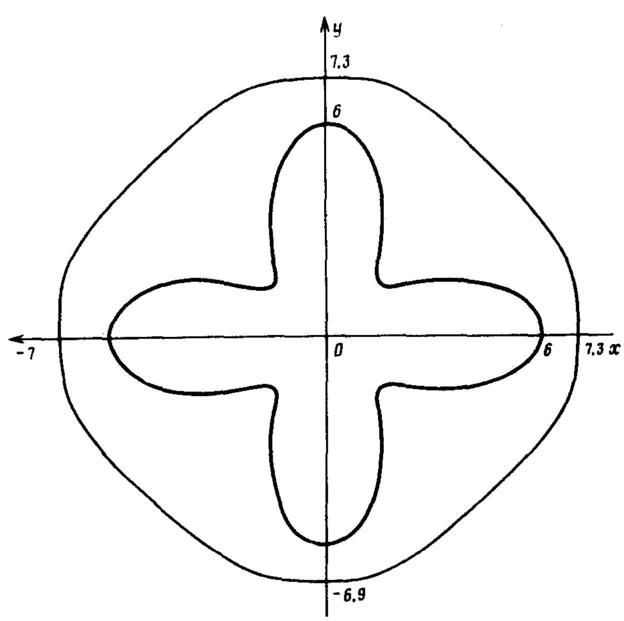

FIG. 1.

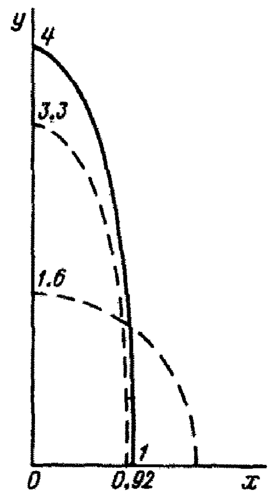

FIG. 2.

As in the three-dimensional case, the problem is non-linear. Nevertheless, the change to the function $P(\alpha)$ reduces the problem to linearity, which is in itself remarkable and apparently not described in the literature. Indeed, we have the relation [4]

$$dx_i+\rho d\alpha_i=0, \quad i=1,\ 2$$

Substituting the expression [4] $x_i = P_i(\alpha)$ we obtain a linear homogeneous system of algebraic equations for $d\alpha_1$ and $d\alpha_2$.

The determinant of this system must vanish, i.e.

$$\rho^2+(P_{11}+P_{22})\rho+P_{11}P_{22}-P_{12}^2=0 \tag{3.2}$$

One can show that the free term in Eq. (3.2) is zero because of the homogeneity properties [4]. We therefore have

$$\rho(\alpha)=-(P_{11}+P_{22}) \tag{3.3}$$

As in the three-dimensional case, in polar coordinates $\alpha_1 = r\cos\theta$, $\alpha_2 = r\sin\theta$ we arrive at the equation

$$P_{\theta\theta}(\theta)+P(\theta)=-\rho(\alpha) \tag{3.4}$$

on the unit circle. The finite-difference approach described above reduces here to a matrix where, as well as a main tri-diagonal band, there is also a non-zero element in each of the upper right and lower left corners.

As examples Figs 2–4 show the results of reconstructions of defect surfaces in the form of ellipsoids with

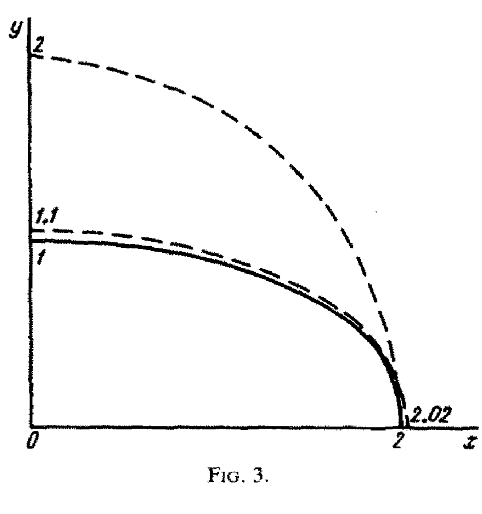

FIG. 3.

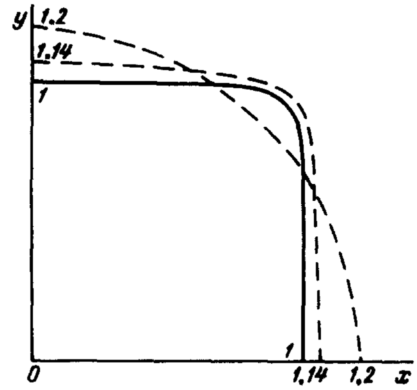

FIG. 4.

semi-axis ratios 1 : 4 and 2 : 1, and also a surface close to a circular cylinder (the base diameter being equal to the height of the cylinder). Because these test examples are for axisymmetric surfaces, only axial sections are shown. Because the axial sections are themselves symmetrical, only the first quadrant is shown. Note that in Figs 2–4 the axis of rotation is the y axis. The broken lines show the first three iterations of the proposed method (the first approximation always being chosen to be a sphere). The solid line shows the actual surface, which in all the examples is nearly identical with the result of the calculation of the third approximation.

REFERENCES

1. SHENDEROV Ye. L., *Wave Problems of Hydro-acoustics*. Sudostroyeniye, Leningrad, 1972.
2. POGORELOV A. V., *The Multidimensional Minkowski Problem*. Nauka, Moscow, 1975.
3. YERMOLOV I. N., *The Theory and Practice of Ultrasonic Monitoring*. Mashinostroyeniye, Moscow, 1981.
4. BLASHKE V., *Differential Geometry and the Geometrical Foundations of Einstein's Theory of Relativity*, Vol. 1. Glav. Red. Obshchetekh. Lit. i Nomgr., Moscow and Leningrad, 1935.
5. PISSANETSKI S., *Sparse Matrix Technology*. Mir, Moscow, 1988.
6. COLTON D. and CRESS R., *Integral Equation Methods in Scattering Theory*. Mir, Moscow, 1987.

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